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Engineering Notes

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Optimal Aircraft Terrain-Following Flight with Nonlinear Engine Dynamics

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I. Introduction

In a recent work on optimal aircraft terrain-following trajectories,[1] it is shown that for a rather general class of aircraft point-mass dynamic models, when the engine dynamics are ignored and the angle of attack is interior, the optimal throttle setting is of bang-bang type in most cases. The optimal throttle setting can be singular only when the terrain is exactly followed and the time-of-flight is not in the performance index. This Note will show that the same conclusion holds when the engine dynamics are modeled by an nth-order nonlinear differential equation. Furthermore, this conclusion still holds true if, in addition, the dynamic characteristics of the engine also depend on Mach number, altitude, and angle of attack.

II. Formulation of Optimal Terrain-Following Problem

Following the problem formulation in Ref. 1, we have the nondimensional equations of planar motion for a point-mass aircraft over a flat Earth

$$h' = V \sin\gamma \tag{1}$$

$$X' = V \cos\gamma \tag{2}$$

$$V' = A_{T\max}\eta \cos\alpha - A_D - \sin\gamma \tag{3}$$

$$\gamma' = (1/V)(A_{T\max}\eta \sin\alpha + A_L) - (\cos\gamma/V) \tag{4}$$

with

$$h = gy/v_s^2, \qquad X = gx/v_s^2, \qquad V = v/v_s \tag{5}$$

$$\tau = gt/v_s, \qquad A_{T\max} = T_{\max}/mg$$
$$A_D = D/mg, \qquad A_L = L/mg \tag{6}$$

where the dimensional variables are the position coordinates x and y, the velocity v, the speed of sound at sea level v_s, the time t, the mass m, the gravitational acceleration g, and the maximum thrust $T_{\max}$. The thrust is assumed to be aligned with the longitudinal body axis, although it is not necessary. The flight path angle is γ. The prime stands for differentiation with respect to τ. L and D are the aerodynamic lift and drag, respectively:

$$L = qSC_L, \qquad D = qSC_D \tag{7}$$

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where S is the reference area of the aircraft, and $q = \rho v^2/2$ is the dynamic pressure with the atmospheric density denoted by ρ that is assumed to be an exponential function of the altitude y with the scale height equal to 23,800 ft. The lift and drag coefficients, C_L and C_D, are functions of the angle of attack α and Mach number M. We will assume for now that the maximum thrust $T_{\max}$ is generally a C^1 function of Mach number and altitude. But we realize that $T_{\max}$ can also be α dependent in high-α flight, which will be addressed later. The thrust percent η is defined to be the ratio between the current thrust level T and $T_{\max}$

$$\eta = T/T_{\max}(M, h) \tag{8}$$

In Ref. 1 η is treated as a control variable. In this Note, we will assume that the engine dynamics, including the lag due to fuel pump and pressure regulator, can be represented by an nth-order nonlinear differential equation of the form

$$\beta\left(\eta, \eta', \ldots, \eta^{(n)}, \eta_{\text{com}}\right) = \eta^{(n)} - a\left(\eta, \eta', \ldots, \eta^{(n-1)}\right)$$
$$- b\left(\eta, \eta', \ldots, \eta^{(n-1)}\right)\eta_{\text{com}} = 0 \tag{9}$$

where η_{com} is the engine throttle setting, and the scalar coefficients a and b are nonlinear functions of their arguments. Then the aerodynamic and propulsion controls are represented by α, in degrees, and η_{com}, respectively, which are subject to the constraints

$$0 \le \eta_{\text{com}} \le 1.0, \qquad -90 < \alpha_{\min} \le \alpha \le \alpha_{\max} < 90 \tag{10}$$

Let the aircraft state vector $\boldsymbol{x} = (h\ X\ V\ \gamma)^T$. The trajectory of the aircraft is to satisfy boundary conditions

$$\boldsymbol{x}(0) = \boldsymbol{x}_0, \qquad \Psi[\boldsymbol{x}(\tau_f)] = 0 \tag{11}$$

where initial conditions are specified. The final time τ_f is free, and $\Psi[\boldsymbol{x}(\tau_f)]$ represents a system of $r \le 4$ algebraic equations that specify the terminal manifold.

Suppose that the terrain is given as a function of the downrange distance X. Let $F(X)$ be the terrain elevation plus the set clearance at X. No matter what the actual terrain is, the function $F(X)$ can always be made sufficiently differentiable. The optimal terrain-following problem is to find the controls $\eta^*_{\text{com}}(\tau)$ and $\alpha^*(\tau)$ such that along a trajectory of the aircraft [Eqs. (1–4) and (9)] satisfying Eqs. (10) and (11), the performance index

$$J = \phi\tau_f + (1 - \phi)\int_0^{\tau_f} [h - F(X)]^2\, d\tau, \qquad 0 < \phi < 1.0 \tag{12}$$

is minimized. It should be pointed out that the second term in Eq. (12) can possibly be zero only in the very special case where the aircraft trajectory at $\tau = 0$ tangentially touches the terrain function $F(X)$. Otherwise, the integral will not be zero.

III. Optimal Bang-Bang Throttle Setting

By the minimum principle,[2] the fact that the throttle setting η_{com} appears linearly in the engine dynamic equation (9) indicates that the optimal $\eta^*_{\text{com}}(\tau)$ either takes on only the upper or lower bound (bang-bang type), or for a finite interval in $[0, \tau_f]$, it continuously varies within the bounds (singular control). Knowing whether or not a singular arc for η^*_{com} exists has a practical significance: it dictates how a numerical algorithm should be implemented to find the optimal solution accurately.[1] Our first result is stated in the following.

Proposition 1. Consider a class of aircraft models for which the lift and drag coefficients take the forms

$$C_L = \sum_{k=1}^{K} C_{L_{2k-1}} \alpha^{2k-1}, \qquad K \ge 1, \qquad C_{L_1} > 0$$
$$C_{L_{2k-1}} \ge 0, \qquad k = 2, \ldots, K \tag{13}$$

$$C_D = \sum_{n=0}^{N} C_{D_{2n}} \alpha^{2n}, \qquad N \ge 0$$
$$C_{D_{2n}} \ge 0, \qquad n = 0, 1, 2, \ldots, N \tag{14}$$

where K and N are positive integers and all of the coefficients $C_{L_{2k-1}}$ and $C_{D_{2n}}$ are C^1 functions of Mach number. The engine dynamics are modeled by Eq. (9) where $a(\cdot)$ and $b(\cdot)$ are C^1. The aircraft state equations are Eqs. (1–4) with m assumed to be constant. The performance index of minimization is

$$J = \phi\tau_f + (1-\phi)\int_0^{\tau_f} [L(h, X)]^2 \, d\tau, \qquad 0 < \phi < 1.0 \tag{15}$$

where $L(h, X)$ is any given function of class C^1, dependent only on h and X. [For the terrain-following problem, $L(h, X) = h - F(X)$. But the Proposition is also valid for other forms of $L(h, X)$.] Assume that an optimal solution satisfying the control constraints (10) and boundary conditions (11) exists, and $b(\eta, \eta', \ldots, \eta^{(n-1)}) \ne 0$ along the optimal trajectory. Then, the optimal throttle setting $\eta^*_{\text{com}}(\tau)$ must be of bang-bang type in any finite interval in $[0, \tau_f]$ where the optimal $\alpha^*(\tau)$ is interior, that is, $\alpha_{\min} < \alpha^*(\tau) < \alpha_{\max}$.

Proof. Define the engine state vector $z = (z_1, \ldots, z_n)^T = (\eta, \ldots, \eta^{(n-1)})^T$. By Eq. (9), we have the engine dynamics

$$z' = f(z) + g(z)\eta_{\text{com}} = \begin{pmatrix} z_2 \\ z_3 \\ \vdots \\ z_n \\ a(z) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b(z) \end{pmatrix} \eta_{\text{com}} \tag{16}$$

Form the Hamiltonian of the system (1–4), (15), and (16),

$$\begin{aligned} H &= p_h V \sin\gamma + p_X V \cos\gamma + p_V (A_{T\max} z_1 \cos\alpha - A_D \\ &\quad - \sin\gamma) + p_\gamma [(1/V)(A_{T\max} z_1 \sin\alpha + A_L) - (\cos\gamma / V)] \\ &\quad + p_z^T f(z) + p_{z_n} b(z) \eta_{\text{com}} + (1-\phi) L^2(h, X) \\ &\triangleq H_1 + p_z^T f(z) + p_{z_n} b(z)\eta_{\text{com}} + (1-\phi)L^2(h, X) \end{aligned} \tag{17}$$

where $p = (p_h \; p_X \; p_V \; p_\gamma)^T$ and $p_z = (p_{z_1}, \ldots, p_{z_n})^T$ are the costate vectors, and p satisfies the adjoint equations $p' = -\partial H / \partial x$. In particular,

$$\begin{aligned} p_V' &= -\frac{\partial H}{\partial V} = -p_h \sin\gamma - p_X \cos\gamma \\ &\quad - p_V \left(z_1 \cos\alpha \frac{\partial A_{T\max}}{\partial V} - \frac{\partial A_D}{\partial V} \right) + \frac{p_\gamma}{V} \left[\frac{1}{V} (A_{T\max} z_1 \sin\alpha \right. \\ &\quad \left. + A_L) - z_1 \sin\alpha \frac{\partial A_{T\max}}{\partial V} - \frac{\partial A_L}{\partial V} - \frac{\cos\gamma}{V} \right] \end{aligned} \tag{18}$$

$$p_\gamma' = -\frac{\partial H}{\partial \gamma} = -p_h V \cos\gamma + p_X V \sin\gamma + p_V \cos\gamma - p_\gamma \frac{\sin\gamma}{V} \tag{19}$$

and p_z satisfies

$$p_z' = -\frac{\partial H}{\partial z} = -\left(\frac{\partial f(z)}{\partial z}\right)^T p_z - \begin{pmatrix} \frac{\partial H_1}{\partial z_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} - p_{z_n} \begin{pmatrix} \frac{\partial b(z)}{\partial z_1} \\ \frac{\partial b(z)}{\partial z_2} \\ \vdots \\ \frac{\partial b(z)}{\partial z_n} \end{pmatrix} \eta_{\text{com}} \tag{20}$$

Since the right-hand sides of the state equations (1–4) and (16), the integrand $L(h, X)$ in the performance index, and the terminal constraint residual Ψ in Eq. (11) all do not depend explicitly on the time, the Hamiltonian is constant along the optimal trajectory[2]

$$H = H_1 + p_z^T f(z) + p_{z_n} b(z) \eta_{\text{com}} + (1-\phi) L^2(h, X) = -\phi \tag{21}$$

Suppose that $[\tau_1, \tau_2] \subset [0, \tau_f]$ is a finite interval where α^* is interior. The optimality condition for α in $[\tau_1, \tau_2]$ requires

$$\begin{aligned} \frac{\partial H}{\partial \alpha} &= -p_V \left(A_{T\max} z_1 \sin\alpha + \frac{\partial A_D}{\partial \alpha} \right) \\ &\quad + p_\gamma \left(A_{T\max} z_1 \cos\alpha + \frac{\partial A_L}{\partial \alpha} \right) \Big/ V = 0 \end{aligned} \tag{22}$$

The proof is completed by contradiction. Now, suppose that a singular arc for η_{com} exists in a finite interval inside $[\tau_1, \tau_2]$. It is, thus, necessary that the switching function for η_{com} be identically zero in this interval, that is, $p_{z_n} b(z) \equiv 0$. Since $b(z) \ne 0$ for all z, we must have $p_{z_n} \equiv 0$. Let us examine the last equation in Eqs. (20),

$$p_{z_n}' = -p_{z_{n-1}} - p_{z_n} \left(\frac{\partial a(z)}{\partial z_n} + \frac{\partial b(z)}{\partial z_n} \eta_{\text{com}} \right) \tag{23}$$

Clearly, with $p_{z_n} \equiv 0$, it is necessary that $p_{z_{n-1}} \equiv 0$ from Eq. (23). Using $p_{z_n} = p_{z_{n-1}} \equiv 0$ in the second to the last equation in the adjoint system (20) gives $p_{z_{n-2}} \equiv 0$. Sequential application of this process up through the second equation in Eqs. (20) produces

$$p_{z_n} = p_{z_{n-1}} = \cdots = p_{z_1} \equiv 0 \tag{24}$$

in this interval. Then, using $p_{z_n} = p_{z_1} \equiv 0$ in the first equation of Eqs. (20) yields

$$\frac{\partial H_1}{\partial z_1} = 0 \Rightarrow p_V \cos\alpha + \frac{p_\gamma \sin\alpha}{V} \equiv 0 \tag{25}$$

One possible solution to Eqs. (22) and (25) is that p_V and p_γ are simultaneously zero in this interval. But $p_V = p_\gamma \equiv 0$ leads from Eqs. (18) and (19) to the system

$$-p_h \sin\gamma - p_X \cos\gamma = 0 \tag{26}$$

$$-p_h V \cos\gamma + p_X V \sin\gamma = 0 \tag{27}$$

which has a unique solution $p_h = p_X = 0$ for $V \ne 0$. Therefore, both costate vectors p and p_z vanish in this interval. Thus, Eq. (21) now reduces to

$$(1-\phi) L^2(h, X) = -\phi \tag{28}$$

Since $0 < \phi < 1$, no values of L can satisfy Eq. (28). We conclude that p_V and p_γ cannot be simultaneously zero for a finite period. This implies that the determinant of the coefficient matrix of the linear algebraic system (22) and (25) in p_V and p_γ must be zero, which for $V \ne 0$ results in

$$A_{T\max} z_1 + \sin\alpha \frac{\partial A_D}{\partial \alpha} + \cos\alpha \frac{\partial A_L}{\partial \alpha} = 0 \tag{29}$$

Since the thrust acceleration $A_{T\max} z_1 \ge 0$, we need

$$\sin\alpha \frac{\partial A_D}{\partial \alpha} + \cos\alpha \frac{\partial A_L}{\partial \alpha} \le 0 \tag{30}$$

for Eq. (29) to hold. Evaluating the partial derivatives in Eq. (30) with C_L and C_D given in Eqs. (13) and (14), and noting that $\cos\alpha > 0$ for $|\alpha| < 90$ deg, we have

$$\begin{aligned} &0.5\left[C_{L_1} + 3C_{L_3}\alpha^2 + \cdots + (2K-1)C_{L_{2K-1}}\alpha^{2K-2}\right] \\ &\quad + \alpha \tan\alpha \left(C_{D_2} + 2C_{D_4}\alpha^2 + \cdots + NC_{D_{2N}}\alpha^{2N-2}\right) \le 0 \end{aligned} \tag{31}$$

Because $\alpha \tan\alpha \ge 0$ for $|\alpha| < 90$ deg, $C_{L_1} > 0$, and all other terms in Eq. (31) are nonnegative, inequality (31) cannot be satisfied by any admissible α. This contradiction rules out the possibility of

$p_{z_n} \equiv 0$ holding in a finite interval inside $[\tau_1, \tau_2]$. It follows that no finite singular arcs exist for the optimal throttle setting in $[\tau_1, \tau_2]$.

One special case of the engine model (9) is a nonlinear first-order lag

$$\eta' = -a(\eta)(\eta - \eta_{\rm com}) \tag{32}$$

where the time-constant $1/a$ depends on the current engine power level specified by η. The engine model for an F-16 aircraft in Ref. 3, for instance, has this feature. Also, a linear nth-order model

$$\eta^{(n)} + a_{n-1}\eta^{(n-1)} + \cdots + a_1\eta' + a_0\eta = a_0\eta_{\rm com} \tag{33}$$

is certainly another special case of Eq. (9), where the a_i coefficients are positive constants such that the polynomial $p(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$ is Hurwitz. [It may be further necessary that $p(s) = 0$ have only negative real roots for Eq. (33) to be a reasonable model, since by definition $0 \le \eta \le 1$, and no overshoot in η from Eq. (33) should exist for $\eta_{\rm com} = 1.0$ or $\eta_{\rm com} = 0$.]

When the weighting ϕ is set to zero in the performance index (15), i.e., the reduction of time-of-flight is not a part of the optimization objective, Eq. (28) can be satisfied when $L(h, X) = h - F(X) = 0$. In other words, singular optimal $\eta_{\rm com}$ can occur only when $\phi = 0$ and the terrain is perfectly followed. A similar conclusion is obtained and a numerical example is given in Ref. 1 for no engine dynamics.

The conclusion in Proposition 1 can be further generalized to include the case where the engine dynamic characteristics are also dependent on Mach number and altitude. Suppose that the engine dynamics are modeled by

$$\eta^{(n)} = a\big(\eta, \ldots, \eta^{(n-1)}, V, h\big) + b\big(\eta, \ldots, \eta^{(n-1)}, V, h\big)\eta_{\rm com} \tag{34}$$

Then we have the following result.

Proposition 2. Assume all of the conditions stated in Proposition 1, except that now the engine model is also dependent on Mach number and altitude as defined by Eq. (34), and $b(\eta, \ldots, \eta^{(n-1)}, V, h) \neq 0$ along the optimal trajectory. Then, the optimal throttle setting $\eta^*_{\rm com}(\tau)$ must be of bang-bang type in any finite interval in $[0, \tau_f]$ where the optimal $\alpha^*(\tau)$ is interior.

Proof. In this case the only change in the optimality conditions used in the proof of Proposition 1 is in the p_V equation,

$$\begin{aligned} p'_V = -\frac{\partial H}{\partial V} &= -p_h \sin\gamma - p_X \cos\gamma \\ &- p_V\left(z_1 \cos\alpha \frac{\partial A_{T\max}}{\partial V} - \frac{\partial A_D}{\partial V}\right) \\ &+ \frac{p_\gamma}{V}\left[\frac{1}{V}(A_{T\max} z_1 \sin\alpha + A_L) - z_1 \sin\alpha \frac{\partial A_{T\max}}{\partial V} - \frac{\partial A_L}{\partial V} \right. \\ &\left. - \frac{\cos\gamma}{V}\right] - p_{z_n}\left(\frac{\partial a(z, V, h)}{\partial V} + \frac{\partial b(z, V, h)}{\partial V}\eta_{\rm com}\right) \end{aligned} \tag{35}$$

where z is the same as defined before. But upon assuming a singular arc for $\eta^*_{\rm com}$, then $p_{z_n} \equiv 0$ and the last term in Eq. (35) drops out. Thus, Eqs. (18) and (35) are identical in this interval. The rest of the proof based on contradiction then follows exactly the proof of Proposition 1.

When the aircraft is in high angle-of-attack flight, the engine characteristics may also be dependent on α. Following a similar proof, we have the following.

Proposition 3 Now assume that $T_{\max}$ is a C^1 function of Mach number, represented by V, altitude h, and angle of attack α, and the engine dynamics are also α dependent,

$$\eta^{(n)} = a\big(\eta, \ldots, \eta^{(n-1)}, V, h, \alpha\big) + b\big(\eta, \ldots, \eta^{(n-1)}, V, h, \alpha\big)\eta_{\rm com} \tag{36}$$

If all of the conditions in Proposition 1 hold and $b(\eta, \ldots, \eta^{(n-1)}, V, h, \alpha) \neq 0$ along the optimal trajectory, then the optimal throttle setting $\eta^*_{\rm com}(\tau)$ must be of bang-bang type in any finite interval in $[0, \tau_f]$ where the optimal $\alpha^*(\tau)$ is interior.

The main difference in the proof in this case is that the optimality condition $\partial H/\partial\alpha = 0$ will have an extra term involving $\partial A_{T_{\max}}/\partial\alpha$ in each of the parentheses following p_V and p_γ as compared to Eq. (22), and two extra terms involving $p_{z_n}\partial a/\partial\alpha$ and $p_{z_n}\partial b/\partial\alpha$. But these terms either become zero because of condition (24), or cancel each other when setting the determinant of the algebraic system in p_V and p_γ to zero. Eventually, a condition exactly the same as Eq. (29) is arrived at that again eliminates the possibility of singular throttle control. The detailed proof is omitted because of space limitations.

As a concluding remark, it should be stressed that although the preceding analysis reveals that bang-bang throttle control is a general property of the optimal terrain-following problem, which requires little specifics on the engine model, the number of throttle setting switchings is expected to depend strongly on the particular engine model assumed.

IV. Conclusions

This Note shows that for a class of nonlinear engine dynamic models of arbitrary order that can also be Mach, altitude, and angle-of-attack dependent, the optimal throttle setting for aircraft terrain-following flight in most cases is of bang-bang type whenever the optimal angle of attack is interior. The only case where a singular throttle control may appear is when the time-of-flight is not included in the performance index and the terrain is followed exactly. The same behavior of the optimal throttle setting is obtained in Ref. 1 where no engine dynamics are included. This result suggests that the usual practice of ignoring engine dynamics in aircraft trajectory optimization work does not lead to incorrect conclusions. Because of the space limitation, the effects of engine dynamics on optimal aircraft terrain-following performance are not explored numerically in this Note, but will be investigated in our future studies.

References

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Long Flight-Time Range-Optimal Aircraft Trajectories

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Introduction

In Ref. 1, indirect solutions based on Pontryagin's minimum principle[2] were obtained for the problem of maximizing the downrange of a high-performance atmospheric flight vehicle operating in the vertical plane. The present Note is based on an identical aircraft model and extends the results obtained in Ref. 1.

Control variables are the load factor that appears nonlinearly in the equations of motion and the throttle setting that appears only linearly. Both controls are subject to fixed bounds. Additionally, a dynamic pressure limit is imposed, which represents a first-order state-inequality constraint.[2]

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